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# One-Dimensional Random Field Ising Model and Discrete Stochastic Mappings

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Previous results relating the one-dimensional random field Ising model to a discrete stochastic mapping are generalized to a two-valued correlated random (Markovian) field and to the case of zero temperature. The fractal dimension of the support of the invariant measure is calculated in a simple approximation and its dependence on the physical parameters is discussed.

**KEY WORDS:** Random field Ising model; stochastic mapping; Markov chains; invariant measure; fractal dimension.

## 1. INTRODUCTION

The calculation of the partition function of the one-dimensional Ising chain in a static random magnetic field can be reduced to the problem of one spin in an auxiliary local random field[1,2]

$$Z = \sum_{\{s_n\}} \exp\left[\beta \sum_{n=1}^{N} (Js_n s_{n+1} + h_n s_n)\right]$$
$$= \sum_{s_N} \exp\left\{\beta\left[\xi_N s_N + \sum_{n=1}^{N} B(\xi_n)\right]\right\} \tag{1}$$

where the local random field $\xi_n$ is governed by the discrete stochastic mapping

$$\xi_n = h_n + A(\xi_{n-1}) = f(h_n, \xi_{n-1}), \qquad \xi_0 = 0, \quad n = 1, 2, \ldots, N \tag{2}$$

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Here

$$A(x) = (2\beta)^{-1} \ln[\text{ch}\, \beta(x+J)/\text{ch}\, \beta(x-J)] \tag{3}$$

$$B(x) = (2\beta)^{-1} \ln[4\, \text{ch}\, \beta(x+J)\, \text{ch}\, \beta(x-J)] \tag{4}$$

The probability density $P(x)$ of the local random field $\xi_n$ can be used to calculate physical quantities such as the free energy, the magnetization, or the Edwards–Anderson parameter.[2]

Obviously, the properties of the stochastic mapping depend on the nature of the driving process $h_n$ and the shape of the function $A$.

For an identical independent distributed two-valued magnetic field it was previously shown[2–5] for nonzero temperatures that for small exchange $J$ the support of $P(x)$ has a fractal structure, whereas for large $J$ the support is continuous. For a continuous distribution the support is the continuum.[5]

In this contribution the previous considerations are extended to a Markovian two-valued magnetic field and to the case of zero temperature. For $T=0$ the support consists of a finite number of points and the theory of finite-state Markov chains is applied to determine the invariant measure. For $T \neq 0$ the fractal dimension of the support is calculated in a simple approximation and its dependence on the physical parameters $(h, J, T)$ is discussed.

## 2. GENERALIZATION TO MARKOVIAN FIELDS

If the external magnetic field $h_n$ is a first-order Markov chain, the auxiliary random field $\xi_n$ is a second-order Markov chain. Introducing the vector $(\xi_n, h_n)$, we have for the Chapman–Kolmogorov equation for the joint probability density $p_n(x, \eta)$

$$p_n(x, \eta) = \int d\eta' \int dx'\, T(\eta \mid \eta')\, p_{n-1}(x', \eta')\, \delta(x - \eta - A(x')) \tag{5}$$

where the transient probability density for the external magnetic field is, e.g.,

$$T(\eta \mid \eta') = \alpha\delta(\eta + \eta') + (1-\alpha)\, \delta(\eta - \eta')$$

$\alpha$ is the probability that $h_n$ changes sign from site $n$ to $n+1$.

For an uncorrelated external field ($\alpha = 1/2$) one finds with $T(\eta \mid \eta') = \rho(\eta)$ the Chapman–Kolmogorov equation for a first-order Markov chain.[2]

For a constant external field ($\alpha = 0$) one reproduces with $T(\eta \mid \eta') = \delta(\eta - \eta')$ and $\rho(\eta) = \delta(\eta - h)$ the fixed-point result $p^*(x, h) = \delta(x - x^*)$, where $x^* = h + A(x^*)$.[1] An alternating external field with period one is obtained if $\alpha = 1$.

The generalization to Markovian fields allows one to interpolate between these limiting cases.

## 3. ZERO-TEMPERATURE PROPERTIES

For zero temperature the function $A(x)$ that governs (2) is piecewise linear,

$$A(x) = \begin{cases} -J & \text{for} \quad x < -J \\ x & \text{for} \quad |x| \leqslant J \\ J & \text{for} \quad x > J \end{cases} \tag{6}$$

As a consequence, for a finite-state driving process, the mapping (2) generates for a given $J$ only a finite number of possible values, so that the fractal dimension of the support at zero temperature is zero.

For an external field taking only the values $\pm h$ the mapping (2) generates only the values

$$x(m, \pm J) = mh \pm J, \qquad x(m, 0) = mh \tag{7}$$

where the integer $m$ has to be chosen such that

$$x \in [h - J, h + J] \cup [-h + J, -h - J] \tag{8}$$

These possible states can be classified into essential and inessential states according to the usual theory of finite-state Markov chains. This classification depends in general on the value of $\alpha$.

The measure consists of a sum of weighted $\delta$-functions located at the points $\{x_i, h_i\}$, which constitute the space of states. Introducing the vector of the weights $\mathbf{w}^{(n)} = \{w_i^{(n)}\}$, one has that the Chapman–Kolmogorov equation (5) converts into the matrix equation

$$\mathbf{w}^{(n)} = D\mathbf{w}^{(n-1)} \tag{9}$$

where the matrix elements of $D$ are $\alpha$ if $x_i^{(n)} = f(h, x_j^{(n-1)} = f(-h, \cdot))$ and $1 - \alpha$ if $x_i^{(n)} = f(h, x_j^{(n-1)} = f(h, \cdot))$, and zero otherwise.

The invariant measure corresponds to the fixed points of (9) given by $(1 - D)\,\mathbf{w}^* = 0$ or by $\mathbf{w}^* = \lim_{n \to \infty} D^n \mathbf{w}^{(0)}$ (if this limit exists). The number of fixed-point solutions is equal to the number of disconnected sets of essential states.

For example, we consider the case $0 < J < h/2$, where we have the flow diagram shown in Fig. 1. For $0 < \alpha < 1$ there are four essential states

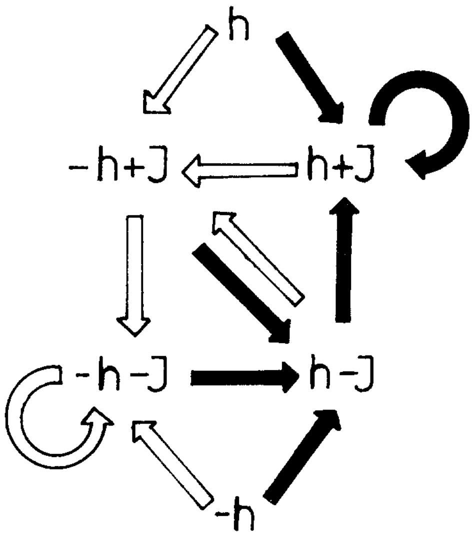


Fig. 1. Flow diagram of the mapping (2) for zero temperature and $0 < J < h/2$. The solid (open) arrows denote the action of (2) for the realizations $h_n = h$ ($h_n = -h$).

$\{(h+J, h),\ (h-J, h),\ (-h+J, -h),\ (-h-J, -h)\}$, which map exclusively into themselves, whereas $h$ and $-h$ are inessential, since there is a net outflow into essential states. The transition matrix between the four essential states is

$$D = \begin{pmatrix} 1-\alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & \alpha & \alpha \\ \alpha & \alpha & 0 & 0 \\ 0 & 0 & 1-\alpha & 1-\alpha \end{pmatrix} \tag{10}$$

The unique fixed point of (9) is $\mathbf{w}^* = (1-\alpha, \alpha, \alpha, 1-\alpha)^T/2$. For $\alpha = 0$, $D$ becomes idempotent and (9) has two different fixed points corresponding to the trapping states $\pm(h+J)$. For $\alpha = 1$ the process oscillates between $-h+J$ and $h-J$ and we observe that $\lim_{n\to\infty} D^n$ does not exist.

For zero temperature and nonzero mean external field $\langle h_n \rangle = h_0$ a similar analysis shows that for $h_0 < h$ the number of states is enlarged compared with the case $h_0 = 0$, whereas for $h \leqslant h_0$ a completely different behavior is found: there are only two essential states, $J + h_0 - h$ and $J + h_0 + h$. It is worthwhile to mention that the latter holds also for the periodic case $\alpha = 1$. A comparison with zero-temperature results obtained by a different method[6] is in preparation.

## 4. NONZERO-TEMPERATURE PROPERTIES

For nonzero temperature $A(x)$ is infinitely many times differentiable and as a consequence (2) generates for $0<\alpha<1$ an infinite number of possible values. These values can be related to infinite sequences of plus and minus signs in the following way.

We denote the result of the $n$th iteration of (2) starting from the initial value $\xi_0 = y$ by

$$x_{\sigma_1,\sigma_2,\ldots,\sigma_n;y} = f(h_1, f(h_2, f(\ldots, f(h_n, y)\cdots)))$$

where $\{\sigma_1,\ldots,\sigma_n\}$ is the sequence of signs of a given realization of the driving process $\{h_1,\ldots,h_n\}$. The result of infinitely many iterations [not depending on the initial value $y$, because of $\sigma_x f(h, x) < 1$] is denoted by $x_{\boldsymbol{\sigma}}$ where $\boldsymbol{\sigma}$ symbolizes an infinite sequence of signs.

Hence, the support of the probability density is the set $S = \{x_{\boldsymbol{\sigma}}\}$, which is an attractor whose basin of attraction is $\mathbb{R}^1$. Any two points $x_{\boldsymbol{\sigma}'}$, $x_{\boldsymbol{\sigma}''} \in S$ can be connected by (2).

It can be shown that starting from an arbitrary initial density $p_0$ the sequence $\{p_n\}$ converges to the unique ergodic invariant measure $p^*$.[7]

For zero mean external field it can be seen by construction that $S \subset [-x^*, x^*]$, where $x^*$ is the fixed point of (2) for $h_n = h$. Obviously, there are parameters for which there are no states $x_{\boldsymbol{\sigma}}$ between the points $x_{+;-\boldsymbol{\sigma}^*} = f(h, -x^*)$ and $x_{-;\boldsymbol{\sigma}^*} = f(-h, x^*)$, i.e., there is a gap of the width (cf. Fig. 2)

$$\Delta = x_{+;-\boldsymbol{\sigma}^*} - x_{-;\boldsymbol{\sigma}^*} = 2(2h - x^*) \tag{11}$$

Applying (2), this gap produces two gaps of the second generation and so on. The two endpoints of one of the $2^{n-1}$ gaps in the $n$th generation can be represented by

$$x_{\sigma_1,\ldots,\sigma_{n-1},+;-\boldsymbol{\sigma}^*} \qquad \text{and} \qquad x_{\sigma_1,\ldots,\sigma_{n-1},-;\boldsymbol{\sigma}^*}$$

We call the finite sequence of $n$ (different) signs $\{\sigma_1,\ldots,\sigma_{n-1}, \pm\}$ the "head" and the infinite sequence of identical signs $\{\mp\boldsymbol{\sigma}^*\}$ the "tail." The set of endpoints is countable. On the other hand, it is dense in $S$: An endpoint is as close to $x_{\boldsymbol{\sigma}}$ as long as its "head" is chosen in such a way that it coincides with the corresponding signs of $\boldsymbol{\sigma}$. Thus, in an arbitrary neighborhood of $x_{\boldsymbol{\sigma}}$ we can find a gap.

Therefore, the support is nowhere dense on $[-x^*, x^*]$ and constitutes a fractal, but it is not self-similar in a simple way like the Cantor set.

Replacing $A(x)$ by $(x^* - h)\, x/x^*$ (cf. dashed lines in Fig. 2), the above procedure gives instead of $S$ the Cantor set $C_\Delta$ with the largest gap equal

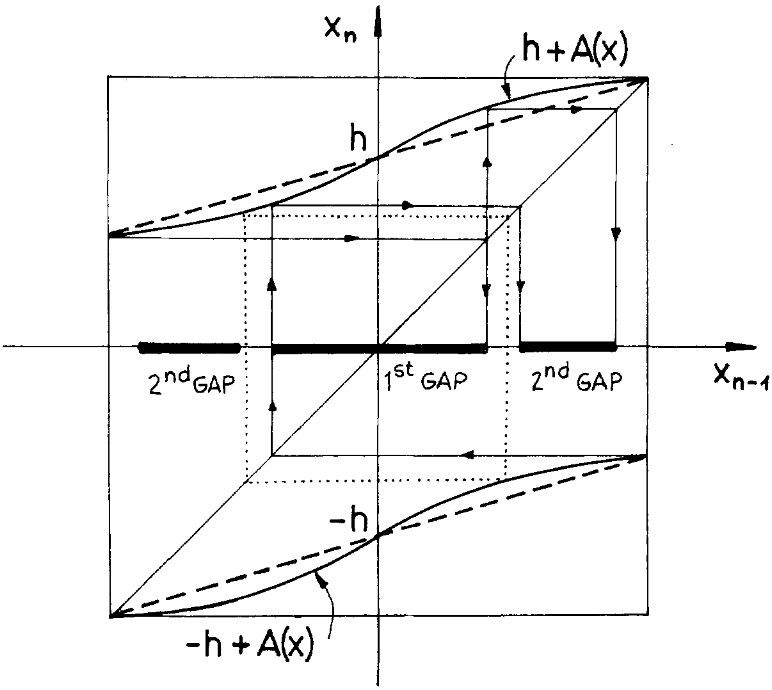


Fig. 2. The construction of the support $S$ of the mapping (2) for nonzero temperature and positive gap. The dashed lines correspond to the Cantor approximation. For an alternating field ($\alpha = 1$), $S$ reduces to an attracting orbit (dotted line).

to $\Delta$. Deviations of $S$ from $C_\Delta$ are due to the nonlinearity of $A(x)$. In this approximation one obtains the fractal dimension (cf. Fig. 3)

$$d_f \approx \begin{cases} 1 & \text{for} \quad \Delta \leqslant 0 \\ \ln 2/\ln[x^*/(x^* - h)] & \text{for} \quad \Delta > 0 \end{cases} \tag{12}$$

The line $\Delta(h, T) = 0$ separates the $(h, T)$ plane into two regions characterized by the fractal dimension of the support (cf. Fig. 4). For zero temperature the support consists of a finite number of points, so that $d_f = 0$. In the gapless region there is a discontinuous transition for $T \to 0$, whereas in the fractal region the transition is continuous. For $T \to \infty$ the fractal dimension also reduces to zero.

Since we are dealing with the one-dimensional Ising model, there are no phases in the thermodynamic sense, but there are parameters such as $d_f$ or the Liapunov exponent which behave as functions of $(h, T, J)$ like "order parameters" and may indicate, e.g., a drastic change in the dynamics.

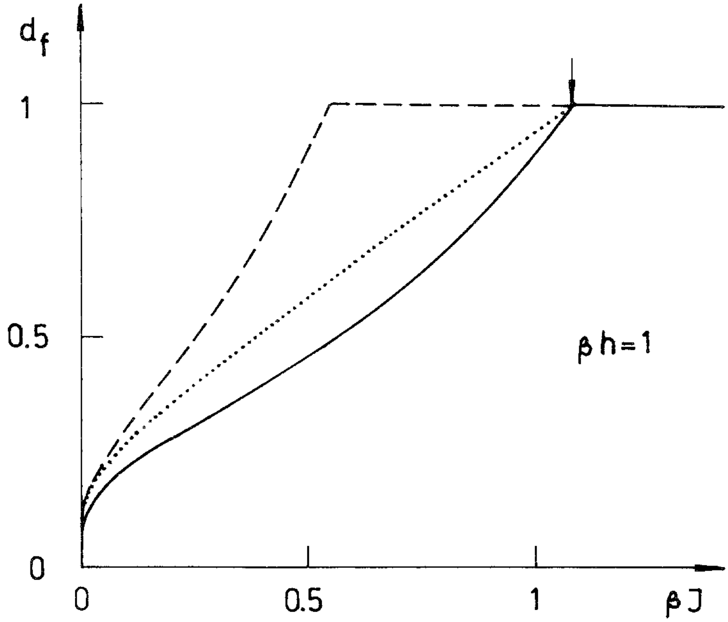


Fig. 3. Fractal dimension versus $\beta J$ for $\beta h=1$ calculated (- -) in zeroth-order perturbation theory,[2] (—) by an iteration procedure,[8] and (···) in the Cantor approximation. The arrow indicates the value of $\beta J$ for which the gap vanishes.

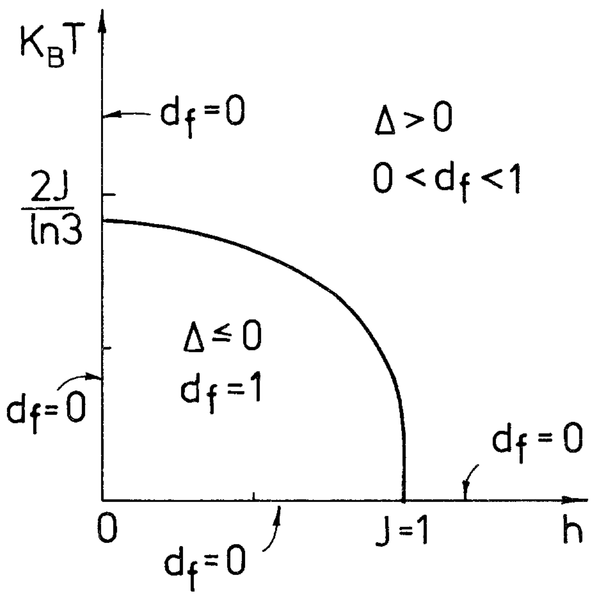


Fig. 4. Qualitative behavior of $d_f$ as a function of temperature and magnetic field for a given $J$.

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